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Statistics of equally weighted random paths on a class of self-similar structures

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Abstract

We study the statistics of equally weighted random walk paths on a family of Sierpinski gasket lattices whose members are labelled by an integer b ($2 \leq b < \infty$). The obtained exact results on the first eight members of this family reveal that, for every $b > 2$, mean path end-to-end distance grows more slowly than any power of its length N . We provide arguments for the emergence of usual power law critical behaviour in the limit $b \rightarrow \infty$ when fractal lattices become almost compact.

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1. Introduction

The statistics of random walk (RW) paths on self-similar structures has attracted a lot of attention over the last two decades [1]. The main focus of related research activities was on the study of large-scale behaviour of various discrete models. As a result of these studies it was recognized that, in the case of RWs on fractals with coordination number that can vary from site to site of the lattice, one can consider several types of statistics (see, e.g., [2] for a nice review).

In particular, in the case of *kinetic* RWs the statistical weight associated with a particular path depends on both the number and type of visited lattice sites. On the other hand, one can associate the same weight K^N with each RW path having N steps, irrespective of the coordination number of visited sites. This model, known also as the *ideal chain* model, is closely related to the equilibrium statistical problem of an ideal polymer in solution. It was shown [3] that the ideal chain model and kinetic RW model in an inhomogeneous environment do not belong to the same class of universality. It is clear, on the other hand, that both statistics become equivalent on standard homogeneous lattices, and on fractal lattices having the same coordination. In view of this, a natural question arises, does a particular statistics of RW paths

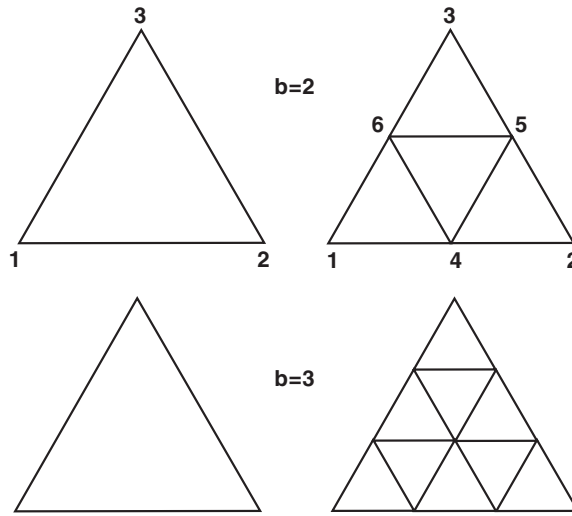


Figure 1. Iterative construction of the first two members of the SG family of lattices: $b = 2$ on the top, and $b = 3$ at the bottom of the figure. The above numbers denote the positions of Gaussian spin variables that we use in (3).

on fractals cross to the same statistics of RW paths on corresponding homogeneous lattices when the underlying self-similar lattice structure becomes almost compact?

This question has challenged a considerable research effort in the near past. Specifically, statistics of self-avoiding RW paths on a family of Sierpinski gasket (SG) lattices, depicted in figure 1, has been much studied (see, e.g., [4–6]). Each member of this family can be labelled by an integer b ($2 \leq b \leq \infty$) so that, when $b \rightarrow \infty$, both fractal and spectral lattice dimensions tend to their homogeneous limit 2. The main conclusion of these studies was that critical exponent ν , which governs the scaling $R_N \sim N^\nu$ of mean end-to-end path distance R_N , should approach its homogeneous limit [5]: $\nu(b) \rightarrow 3/4$, when $b \rightarrow \infty$.

The problem of kinetic RW paths on a fractal lattice is known to be closely related to linear dynamics [7] of the same lattice. Thus the mean path end-to-end distance critical exponent ν is shown to depend only on fractal and spectral dimension of the underlying self-similar structure. Both fractal and spectral dimensions of the above-mentioned SG lattice family have also been studied [8–11]. Using the results of these studies, one can conclude that in the limit of large values of the lattice parameter b normal diffusion [7] emerges: $\nu(b) \simeq 1/2 + \ln[\ln(b)]/(4 \ln b)$, for $b \gg 1$.

In this paper we shall study statistics of equally weighted RW paths (to be referred to as RW paths) on the same class of lattices. As we mentioned above, this statistics can be very different from the statistics of kinetic RW paths. For example, it has been shown [3] that, under certain conditions, the mean end-to-end path distance R_N can grow more slowly than any power of its length N . This effect has been termed localization [3], and it was attributed to entropic trapping: in order to maximize its entropy it is more beneficial for a path to visit the sites of higher coordination number than those with smaller coordination. The sites of highest coordination number act, in fact, as entropic traps which prevent the swelling of the paths.

To study the statistics of RW paths we use a renormalization group method which relies on a simple Gaussian model (see [12], or [3] for a similar approach). We present exact results for the first eight members of the SG family ($2 \leq b \leq 9$). Although these results reveal that

the above-mentioned path localization takes place for every finite $b > 2$, we shall argue that the usual power law critical behaviour should emerge in the limit $b \rightarrow \infty$.

Our model is presented in the following section, where we also outline our approach, recall some previously obtained results and give details of new analytical and numerical results. An overall discussion of these results and conclusion will be given in section 3.

2. Model and its analysis

It is well known that statistics of random-walk paths on lattices can be captured by using a suitable Gaussian model (see, for instance, [13]). For example, the two-point correlation function of a Gaussian model provides the generating function for the numbers of RW paths between two given points, while associated correlation length ξ provides a measure of the average path size. It is expected, therefore, that ξ follows the scaling behaviour of the mean path end-to-end distance R_N .

To outline our approach, and to introduce the notation, we consider firstly the simple Gaussian model on the SG lattice of base $b = 2$.

2.1. $b = 2$, power law behaviour

The partition function of this model on a lattice at the r th stage of its iterative construction has the following form:

$$\mathcal{Z}^{(r)}(K) = \int_{-\infty}^{\infty} dS_1 \cdots dS_N \exp \left[-\frac{1}{2} \sum_i S_i^2 + K \sum_{\langle ij \rangle} S_i S_j \right], \quad (1)$$

where S_i is the continuous spin variable at site i , K stands for the interaction between each nearest-neighbour pair of spins and $\langle ij \rangle$ denotes the summation over all such pairs.

Let us imagine that we have performed the integration over all internal spins (except over three corner spins, see figure 1) of an r th-order triangle. It is clear then that resulting 'partial' partition function should take the form

$$Z_{1,2,3}^{(r)} = D^{(r)} \exp \left[A^{(r)} (S_1^2 + S_2^2 + S_3^2) + B^{(r)} (S_1 S_2 + S_1 S_3 + S_2 S_3) \right], \quad (2)$$

where S_1 , S_2 and S_3 denote three corner spins, while parameters $A^{(r)}$, $B^{(r)}$ and $D^{(r)}$ depend on the iteration index r and interaction K . These parameters can be determined recursively. Indeed, the r th-order partition function (2) can be obtained from similar $(r-1)$ th-order partial partition functions,

$$Z_{1,2,3}^{(r)} = \int dS_4 dS_5 dS_6 Z_{1,4,6} Z_{4,2,5} Z_{5,6,3} \exp \left(-\frac{S_4^2 + S_5^2 + S_6^2}{2} \right) \quad (3)$$

where S_4 , S_5 and S_6 denote three internal corner spins of the r th-order triangle (see figure 1), while $Z_{1,2,3}^{(r)}$ and $Z_{1,4,6}$, for instance, stand for $Z_{1,2,3}^{(r)}$ and $Z_{1,4,6}^{(r-1)}$, respectively. In this way we find the following recursion relations:

$$A' = f_1(A, B) = \frac{A - 8A^2 + 16A^3 - AB + 4A^2B + B^2 - 6AB^2}{(1 - 4A + B)(1 - 4A - 2B)} \quad (4)$$

$$B' = f_2(A, B) = \frac{B^2(1 - 4A + 2B)}{(1 - 4A + B)(1 - 4A - 2B)} \quad (5)$$

and a similar relation for D' which we do not need here. These relations have to be supplemented with the initial conditions: $A^{(0)} = 0$, $B^{(0)} = K$.

A numerical analysis reveals that for all values of K less than $K_c = 1/4$, under iterations of the above system, $B^{(r)}$ rapidly decreases to zero, while $A^{(r)}$ approaches a certain K -dependent constant. On the other hand, if K is larger than the threshold value K_c , both $B^{(r)}$ and $A^{(r)}$ display some spurious instabilities, which reflect the appearance of an unphysical region (the Gaussian model is not defined in the low-temperature region). Finally, precisely at $K = K_c$ all successive iterations of $B^{(r)}$ and $A^{(r)}$ lie on an invariant line, which starts at the point $(A^{(0)} = 0, B^{(0)} = K_c)$ and ends at the fixed point $(A^* = 1/4, B^* = 0)$. It is easy to see, however, that the latter point is singular, which does not allow us to make a common fixed-point analysis. Numerical analysis also shows that this point represents a sink for all points lying on the invariant line. To determine the proper form of recursion relations in the neighbourhood of this point it is useful to know at least an asymptotic equation of the line. As detailed in [12], the asymptotic equation of such a line can be determined perturbatively, by using the above recursion relations and a functional equation of the line having the following general form:

$$f_1[f(B), B] = f[f_2(f(B), B)] \quad (6)$$

where $A = f(B)$ denotes the equation of the requisite line. In spite of the fact that the above functional equation does not look very simple, it allows a simple closed form solution

$$A = f(B) = 1/4 - B \quad (7)$$

which provides the exact equation of the invariant line. Taking into account (5), we find that along this line parameter B decreases much more slowly

$$B' = \frac{3}{5}B \quad (8)$$

which also implies a slow falling down of two-spin correlations. Indeed, it is a simple matter to show that the correlation function between two corner spins of an r th-order triangle, or generating function G for the numbers of paths between two corners, can be expressed in terms of A and B ,

$$G^{(r)} = \frac{B^{(r)}}{1 - 4A^{(r)} + 4A^{(r)2} - B^{(r)} + 2A^{(r)}B^{(r)} - 2B^{(r)2}} \quad (9)$$

which means that the large-scale behaviour of $G^{(r)}$ is governed by $A^{(r)}$ and $B^{(r)}$. In what follows we shall focus on the asymptotic behaviour of $A^{(r)}$ and $B^{(r)}$.

In particular, the number r_0 of iterations along the invariant line, for $K \neq K_c$, depends on the value of $\delta K = K_c - K$ ($0 < \delta K \ll 1$) and can be estimated from the obvious relation,

$$B^{(r_0)}(\delta K) \simeq B^{(r_0)}(0) + \left. \frac{dB^{(r_0)}}{dK} \right|_{\delta K=0} \delta K. \quad (10)$$

Derivative $B_K^{(r_0)} = dB^{(r_0)}/dK|_{\delta K=0}$, together with an analogous derivative of $A^{(r_0)}$ with respect to K , can be calculated iteratively

$$\begin{pmatrix} A_K^{(r_0)} \\ B_K^{(r_0)} \end{pmatrix} = \prod_{r=1}^{r=r_0} \begin{pmatrix} \frac{\partial A^{(r)}}{\partial A^{(r-1)}} & \frac{\partial A^{(r)}}{\partial B^{(r-1)}} \\ \frac{\partial B^{(r)}}{\partial A^{(r-1)}} & \frac{\partial B^{(r)}}{\partial B^{(r-1)}} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (11)$$

which allows us to find the asymptotic behaviour of the requisite derivatives. One can show that for $b = 2$ elements of the above matrices do not depend on r (for large r and $K = K_c$) which implies

$$A_K^{(r_0)} \sim B_K^{(r_0)} \sim \lambda_0^{r_0} \quad (12)$$

where $\lambda_0 = 3$ represents the larger eigenvalue of the above matrices ('thermal' eigenvalue in the context of standard renormalization group terminology).

Taking into account (8), i.e. $B^{(r_0)} \sim (3/5)^{r_0}$ along the invariant line, and assuming that the two terms on the right-hand side of (10) are of the same order of magnitude, we obtain $\delta K \sim 5^{-r_0}$. On the other hand, away from the critical line (see (5)) we have $B^{(r)} \sim \mu^{2^r}$, where $\mu = \mu(\delta K) < 1$. These two regimes are separated by a narrow crossover region, which makes it possible to apply an asymptotic matching [14]. In this way we obtain $\xi = -(\ln[\mu(\delta K)])^{-1} \sim 2^{r_0}$, which is expected on the grounds of finite-size scaling arguments. Taking into account these results, it is easy to show that in this case the correlation length follows the well-known (see, e.g., [16]) power law critical behaviour

$$\xi \sim N^{\ln 2 / \ln 5} \quad (13)$$

where N is the average number of RW steps which present a variable conjugated to δK .

2.2. $b = 3$, strong localization

This case was first studied by Maritan [3] in the context of localization of an ideal polymer chain by entropic trapping, due to the inhomogeneities in the lattice coordination number. As a consequence of this, the mean end-to-end distance of the RW path follows a logarithmic rather than power law behaviour. Description of this critical behaviour has been further deepened and slightly corrected by a study of oscillations of critical amplitudes of the correlation length [15]. In particular, using an asymptotic matching technique, similar to that outlined in the preceding subsection, it has been shown that ξ follows the law

$$\xi \sim \ln^\Phi(N) \quad \text{with} \quad \Phi = \frac{\ln(3/2)}{\ln 2} \quad (14)$$

which differs (in the value of Φ) from the results [3, 2, 12] which have been derived by using the standard finite-size scaling assumption: $\xi \sim 3^{r_0}$, implying that $\Phi = \ln 3 / \ln 2$. To avoid possible difficulties, we preferred to use the method of asymptotic matching in this paper. Let us note, however, that in this respect the $b = 3$ case is exceptional: for all other values of b the results derived by using the asymptotic matching technique are in agreement with those which rely on the finite-size scaling assumption (i.e. we found $\xi \sim b^{r_0}$, if $b \neq 3$).

2.3. $b \geq 4$, weak localization

The two examples presented so far are not generic, in the sense that the RW paths studied in section 2.1 exhibit the usual power law behaviour, while those studied in section 2.2 are strongly localized. On the other hand, for all $b \geq 4$ the RW paths are also localized—they grow more slowly than any power of N (or $1/\delta K$), but more strongly than (14). For this reason we term such a path localization as weak.

Now we are going to present our results for the asymptotic behaviour of RW paths on the lattices with parameter $b \geq 4$. We have been able to generate the exact recursion relations for parameters A and B on the lattices with $4 \leq b \leq 9$. These relations are, however, rather cumbersome and will be omitted here (they are available on request). Instead, we give here only the asymptotic behaviour of some quantities introduced in section 2.1. Thus, in the case $b = 4$, for $K = K_c = 0.213\ 115\dots$, we have found that the equation of the invariant line has the form

$$A = f(B) = c_0 + c_1 B + c_2 B^2 + c_3 B^3 + O(B^4) \quad (15)$$

where coefficients $c_0 = 1/6$, $c_1 = -1/3$, $c_2 = -3$ and $c_3 = -8$ have been determined from a system of nonlinear equations. This system has, in fact, a large number of different solutions and to select the proper one we first roughly estimated the above coefficients by a numerical iteration of recursion relations for $K = K_c$.

Along the invariant line the recursion relation for parameter B acquires the simple form

$$B' = \frac{2}{5}B - \frac{62}{25}B^2 + O(B^3) \quad (16)$$

which allows us to find the leading and next-to-leading (multiplicative) terms in the asymptotic expansion of derivatives in (11),

$$A_K^{(r_0)} \sim B_K^{(r_0)} \sim \left(\frac{5}{2}\right)^{r_0} \lambda_1^{-r_0} \quad \text{where } \lambda_1 = 1.711\,590\dots \quad (17)$$

Let us note here that this form of the asymptotic behaviour is quite different from (12), which is caused mainly by the fact that corresponding matrices appearing in (11) now depend on the iteration index r (matrix elements grow typically as $(5/2)^r$ for $r < r_0$).

Taking into account the above result, it is easy to express the leading asymptotic behaviour of the correlation length: $\xi \sim \exp[\ln 4 \sqrt{\ln(N)}/\sqrt{\ln(5/2)}]$, which states precisely the meaning of the term ‘weak localization’ that we used above.

For all other values of the lattice parameter b we have found basically a similar leading asymptotic behaviour. In order to determine an asymptotic equation of renormalization of parameter B we need typically the first three terms on the right-hand side of (15). It turns out that coefficient c_0 always takes the same value $c_0(b) = 1/6$ ($b \geq 3$), while all values of the coefficient c_1 satisfy the following function:

$$c_1(b) = -\frac{1 + 2 \cos(2\pi/b)}{3} \quad b \geq 2. \quad (18)$$

Let us mention in passing that for $b = 3$ we have $c_1 = 0$, which implies that the expansion of the renormalization equation for B' starts with a quadratic term per B rather than with a linear term (compare (16)). This provides, in fact, the mathematical mechanism of ‘strong localization’ (14).

For $b \geq 4$ we have always found a form of the type (16)

$$B' = \frac{1}{\lambda_0}B + O(B^2) \quad (19)$$

where λ_0 depends on the values of b . On the other hand, the asymptotic behaviour of derivatives $A_K^{(r_0)}$ and $B_K^{(r_0)}$ is found to be generally more complex than (17), and can be described in the following way:

$$B_K^{(r_0)} \sim \lambda_0^{cr_0} \lambda_1^{r_0/2} \lambda_2^{r_0/2} \quad r_0 \gg 1 \quad (20)$$

where constants c , λ_1 and λ_2 depend on the lattice parameter b and can be determined exactly. In contrast to the case $b = 4$, for some other values of b matrix elements of matrices appearing in (11) can be negative, which typically leads to the negative values of the above growth constants λ_1 and λ_2 . As a consequence, in this case matrix product (11) has an oscillatory character which makes its asymptotic analysis more delicate. To check the asymptotic form (20) numerically, we determined the critical values K_c very accurately. This enables us to come very close to the critical point ($\delta K/K_c < 1/10^{1000}$) and thus to confirm (20) numerically. As numerical analysis also reveals, for an accurate estimate of the number r_0 of iterations along the invariant line one has to consider the next-to-leading asymptotic term of this form, in addition to the leading one, especially for larger values of the lattice parameter b . Asymptotic behaviour of $A_K^{(r_0)}$ can also be described by the form (20), with possible different values of λ_1 and λ_2 .

Taking into account these results, together with the above-mentioned general result $\xi \sim b^{r_0}$, it can be verified that the leading asymptotic behaviour of ξ depends only on λ_0 and c ,

$$\xi \sim \exp \left[\frac{\ln b}{\sqrt{c \ln \lambda_0}} \sqrt{\ln(N)} \right]. \quad (21)$$

Table 1. Values of critical interaction strengths (fugacities) K_c and parameters entering asymptotic law (20).

b	K_c	λ_0	λ_1	λ_2	c
4	0.213 115 350	2.5	0.584 251 965	0.584 251 965	1
5	0.203 070 314	1.927 050 983	-0.418 908 482	-0.418 908 482	2
6	0.195 670 774	1.75	$1.477\ 328\ 104 \times 10^3$	$1.477\ 328\ 104 \times 10^3$	1
7	0.190 148 793	1.667 562 802	$6.683\ 130\ 874 \times 10^4$	$6.683\ 130\ 874 \times 10^4$	1
8	0.185 984 832	1.621 320 343	$-2.543\ 952\ 422 \times 10^6$	$-1.618\ 126\ 100 \times 10^7$	2
9	0.182 803 196	1.592 396 265	$8.244\ 728\ 644 \times 10^4$	$8.244\ 728\ 644 \times 10^4$	1

Nevertheless, we believe that the growth constants which enter into the next-to-leading asymptotic terms of $A_K^{(r_0)}$ and $B_K^{(r_0)}$ are useful for a more precise description of the correlation length, in particular for $b \gg 1$. Thus, in addition to the values of the constants λ_0 and c , we give also λ_1 and λ_2 (see table 1).

3. Discussion and summary

In this paper we have studied the statistics of RW paths on the SG family of lattices. We have found three different types of critical behaviour. For $b = 2$ all lattice sites have the same coordination number (4) which leads to the standard power law behaviour (13). For all other values of b a subtle interplay between the sites of coordination 4 and those with coordination number 6 takes places: in the statistics of equally weighted trajectories it is profitable for a path to visit the sites of higher coordination number preferentially. This gives rise to a localization of RW paths on the lattices with $b \geq 3$. This effect is most pronounced for $b = 3$ ('strong localization'), in which case a fraction of lattice sites with coordination number 6 has a minimum. Let us mention, moreover, that in the $b = 3$ case the sites with coordination 6 are isolated, while for $b > 3$ they are grouped into isolated islands, comprised of $(b-2)(b-1)/2$ sites.

The fraction $(b-2)/(b+4)$ of lattice sites with coordination 6 increases with an increase in the lattice parameter b , which leaves more freedom to the RW paths to realize the configurations they prefer. It is expected, therefore, that an increase in the lattice parameter b facilitates the swelling of the paths and thus makes them less localized. This is in accord with the general form (21) of our correlation length and data presented in table 1 (i.e. ξ increases with b).

Here we have presented exact results for only the first eight members of the SG lattice family. It is tempting to speculate what can happen for larger values of the lattice parameter b .

In the limit of large b our lattices converge to a compact triangular lattice. Thus one should recover the usual power law behaviour in the limit $b \rightarrow \infty$. At first glance this is in contrast with the expectation that general formula (21) should be valid for every finite $b \geq 4$. Let us note, however, that parameter λ_0 , which, together with c and b , governs the asymptotic law (21), decreases with b (see table 1). It seems, in fact, that $\lambda_0 \rightarrow 1$ from above for large b , which provides a sign of the inapplicability of (21) in the limit $b \rightarrow \infty$. Indeed, a rough estimate of $\lambda_0(\infty)$, based on the above quoted sequence $\lambda_0(b)$, indicates that $\lambda_0(\infty)$ should be close to 1 for large b . This makes us believe that for sufficiently large values of b , there exists a range of values in the critical region δK where ξ follows a power law. This power law will be determined by λ_1 and λ_2 , providing the general form (20) of the asymptotic behaviour of $A_K^{(r_0)}$ and $B_K^{(r_0)}$ persists for large b (note that (20) acquires form (12) for λ_0 close to 1). It is

expected that, for large but finite b , this power law crosses to the true asymptotic law (21) for sufficiently long RW paths, $N \gg 1$ (i.e., for sufficiently small δK). This, in particular, points out the necessity of exerting much care when interpreting results of numerical simulation: for example, if the RW paths are not sufficiently long, one can easily come to an erroneous conclusion that their critical behaviour is of the power law type.

The width of the region where asymptotic behaviour (21) sets in depends on the ratio of the leading and next-to-leading term in (20). One can note that this ratio generally decreases when the lattice parameter b increases—compare the values of the coefficients λ_0 , λ_1 and λ_2 for different values of b in table 1. As a consequence of this, the power law critical behaviour becomes valid over a wider and wider critical region when b grows. Finally, in the $b \rightarrow \infty$ limit the power law should be valid in the entire critical region, which corroborates our picture of crossover from localized to extended path trajectories.

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